WKB solution of the wave equation for a plane angular sector

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The two angular Lamé differential equations that satisfy boundary conditions on a plane angular sector (PAS) are solved by the Wentzel-Kramers-Brillouin (WKB) method. The WKB phase constants are derived by matching the WKB solution with the asymptotic solution of the Weber equation. The WKB eigenvalues and eigenfunctions show excellent agreement with the exact eigenvalues and eigenfunctions. It is shown that those WKB eigenvalues and eigenfunctions that contribute substantially to the scattering amplitude from a PAS can be computed in a rather simple way. An approximate formula for the WKB normalization constant, which is consistent with the WKB assumptions, is derived and compared with the exact normalization constant. [S1063-651X(98)00507-8]

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I. INTRODUCTION

In a previous paper [1], formulas for the Wentzel-Kramers-Brillouin (WKB) eigenvalues satisfying Dirichlet or Neumann boundary condition on a plane angular sector (PAS) were reported and the WKB eigenvalues, $\{\nu,\mu\}$, were compared with the exact eigenvalues for PAS's of different corner angles (60°, 90°, and 120°). A historical review of the solution of the wave equation for a PAS was also given in the above paper. It suffices to say that, to our knowledge, no approximate solution of this problem has been reported in the literature.

In this paper, the two coupled Lamé equations are solved by the WKB method. The WKB analysis in this paper is valid for large values of ν. Depending on the sign of μ, one of the two Lamé equations has turning points. When $|\mu|$ is small the turning points occur where the angles are small. In this region it proves more accurate to obtain the WKB phase constants by matching it with the asymptotic solution of the Weber equation. For large values of $|\mu|$, it is shown that this phase constant reduces to the phase constant obtained when the solution is matched with the asymptotic solution of the Airy's equation as is commonly done in quantum mechanics.

This paper is organized in the following way: In the second section the WKB solution is formulated and the WKB phase constants are derived. In the third section formulas for the WKB eigenvalues for Dirichlet and Neumann boundary conditions are derived and a comparison between the WKB and exact eigenfunctions is presented. The fourth section contains a derivation of approximate WKB solutions which are valid for small values of μ/ν. Finally, an approximate formula for the WKB normalization constant consistent with WKB assumptions is derived in section five.

II. THE GENERAL SOLUTION

The angular part of the wave equation in the sphero-conal coordinate system are expressed by the two angular Lamé differential equations [2,1]:

$$\sqrt{1-\kappa^2\cos^2\vartheta}\,\frac{d}{d\vartheta}\left[\sqrt{1-\kappa^2\cos^2\vartheta}\,\frac{d}{d\vartheta}\Theta(\vartheta)\right] + [\nu(\nu+1)\kappa^2\sin^2\vartheta+\mu]\Theta(\vartheta)=0 \tag{1}$$

and

$$\sqrt{1-\kappa'^2\cos^2\varphi}\,\frac{d}{d\varphi}\left[\sqrt{1-\kappa'^2\cos^2\varphi}\,\frac{d}{d\varphi}\Phi(\varphi)\right] + [\nu(\nu+1)\kappa'^2\sin^2\varphi-\mu]\Phi(\varphi)=0, \tag{2}$$

where the sphero-conal coordinate system variables ϑ and φ are related to the Cartesian coordinate variables, x, y, and z by

$$x=r\,\cos\,\vartheta\sqrt{1-\kappa'^2\cos^2\varphi},$$

$$y=r\,\sin\,\vartheta\,\sin\,\varphi, \tag{3}$$

$$z=r\,\cos\,\varphi\sqrt{1-\kappa^2\cos^2\vartheta},$$

where

$$\kappa=\cos\left(\frac{\beta}{2}\right),$$

β is the corner angle and $\kappa'=\sqrt{1-\kappa^2}$. The range of the variables is

$$0\leqslant\vartheta\leqslant\pi,\quad 0\leqslant\varphi\leqslant 2\pi,\quad r\geqslant 0.$$

The geometry of the sphero-conal coordinate system is shown in Fig. 1. The construction of this coordinate system is described, and its orthogonality proved, in [2]. The WKB solution of Eqs. (1) and (2) are respectively given by [1]

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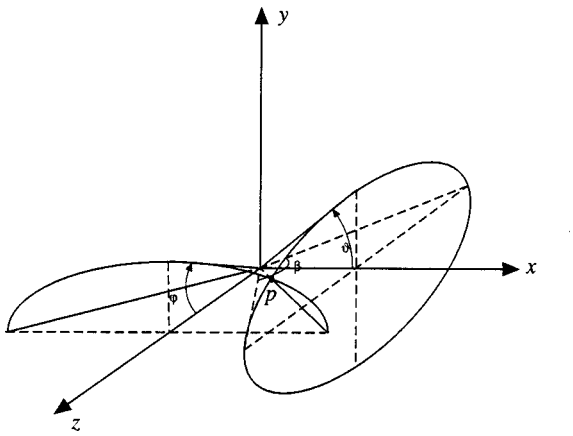

FIG. 1. The geometry of the sphero-conal coordinate system. r is the distance from the origin to the point p.

$$\Theta(\vartheta)=\frac{1}{\sqrt[4]{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}}\times\cos\left\{\int_{\vartheta_t}^{\vartheta}\left(\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{1-\kappa^2\cos^2\vartheta}\right)^{1/2}d\vartheta+\delta_\vartheta\right\}, \tag{4}$$

and

$$\Phi(\varphi)=\frac{1}{\sqrt[4]{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}}\times\cos\left\{\int_{\varphi_t}^{\varphi}\left(\frac{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}{1-\kappa'^2\cos^2\varphi}\right)^{1/2}d\varphi+\delta_\varphi\right\}. \tag{5}$$

Notice that the Φ solution can be obtained from the Θ solution by replacing ϑ with φ, μ with $-\mu$, and κ with κ'. This is also evident in the differential equations, Eqs. (1) and (2). For $\mu>0$, the turning point for the Θ equation, $\vartheta_t=0$, and the turning point for the Φ equation is $\varphi_t=\phi_t(\nu,-\mu,\kappa')$. For $\mu<0$, $\varphi_t=0$ and $\vartheta_t=\phi_t(\nu,\mu,\kappa)$, where

$$\phi_t(\nu,\mu,\kappa)=\arcsin\left(\frac{-\mu}{\kappa^2(\nu+\frac{1}{2})^2}\right)^{1/2}.$$

The above solutions are valid for $\pi-\vartheta_t>\vartheta>\vartheta_t$ and $\pi-\varphi_t>\varphi>\varphi_t$. For the regions $\vartheta_t>\vartheta>\pi-\vartheta_t$ and $\varphi_t>\varphi>\pi-\varphi_t$ the solutions are given by

$$\Theta(\vartheta)=\frac{1}{\sqrt[4]{|\mu|-(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta}}\times\exp\left\{-\int_0^{\vartheta}\left(\frac{|\mu|-(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta}{1-\kappa^2\cos^2\vartheta}\right)^{1/2}d\vartheta\right\},\quad \mu<0. \tag{6}$$

A similar solution for the Φ equation which is valid for $\mu>0$ can be obtained from the above equation by replacing $|\mu|$, κ, and ϑ with μ, κ', and φ, respectively. The solutions given by Eqs. (4) and (6) are separated by the turning points where they are both singular. To match these solutions at the turning points, the Lamé equation is approximated by another differential equation at the turning point and then the asymptotic solution of this differential equation is matched with the WKB solution of the Lamé equation. By this process the phase constants δ_ϑ and δ_φ are obtained.

Although the WKB solutions obtained in this paper are general, we are particulary interested in those solutions that correspond to the small absolute values of the eigenvalue μ. The reason for this is that the expression for the scattering amplitude for a PAS, derived in a separate paper [3], contains either the eigenfunctions or their derivatives evaluated at the surface of the PAS or its edges. Observe that the surface of the PAS in the sphero-conal coordinate system is $\vartheta=\pi$ and its edges are $\varphi=0$ and $\varphi=\pi$. Since the WKB eigenfunctions are decaying exponentials to the left of the turning point near zero and to the right of the turning point near π, significant contribution to the solution comes from those eigenfunctions that have a turning point near the surface or near the edges. They correspond to eigenfunctions with small absolute values of the eigenvalue μ.

To derive a differential equation that approximates the Lamé equation at the turning points, we use the following transformation:

$$v(\vartheta)=\sqrt[4]{1-\kappa^2\cos^2(\vartheta)}\,\Theta(\vartheta).$$

For $\nu\gg\mu\gg1$ this transformation converts Eq. (1) to

$$\frac{d^2}{d\vartheta^2}v(\vartheta)+p(\vartheta)v(\vartheta)=0, \tag{7}$$

with

$$p(\vartheta)=\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{(1-\kappa^2\cos^2\vartheta)}.$$

By using the Liouville transformation [4]

$$y(x)=\left(\frac{dx}{d\vartheta}\right)^{1/2}v(\vartheta),$$

Eq. (7) can be transformed to

$$\frac{d^2y}{dx^2}=\left\{-\left(\frac{d\vartheta}{dx}\right)^2p(\vartheta)+\left(\frac{d\vartheta}{dx}\right)^{1/2}\frac{d^2}{dx^2}\left[\left(\frac{d\vartheta}{dx}\right)^{-1/2}\right]\right\}y.$$

The first term in the curly brackets can be set equal to any smooth function of x [4], and for small μ/ν the second term can be ignored. Near the turning points ($\vartheta\ll1$) we set

$$\left(\frac{d\vartheta}{dx}\right)^2p(\vartheta)=\left(\frac{x^2}{4}+a\right), \tag{8}$$

resulting in

$$\frac{d^2y}{dx^2}+\left(\frac{x^2}{4}\pm a\right)y=0, \tag{9}$$

which is the Weber equation. In the above equation the plus sign is applicable to the Θ equation and the minus sign is applicable to the Φ equation. The parameter a is determined from Eq. (8) by requiring that the turning points of the Lamé equation and the Weber equation occur at the same time, thus ensuring the regularity of $d\vartheta/dx$ at the turning points:

$$\int_0^{\vartheta_t}\left(\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{1-\kappa^2\cos^2\vartheta}\right)^{1/2}d\vartheta=\int_0^{2\sqrt{-a}}\left(\frac{x^2}{4}+a\right)^{1/2}dx. \tag{10}$$

A similar equation is obtained for the Φ equation by replacing ϑ_t by φ_t, κ by κ', μ by $-\mu$, and a by $-a$. Once a is determined from the above equation, ϑ and φ can be related to x by

$$\int_0^{\vartheta_t}\left(\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{1-\kappa^2\cos^2\vartheta}\right)^{1/2}d\vartheta=\int_0^{x}\left(\frac{x^2}{4}+a\right)^{1/2}dx. \tag{11}$$

and

$$\int_{\varphi_t}^{\varphi}\left(\frac{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}{1-\kappa'^2\cos^2\varphi}\right)^{1/2}d\varphi=\int_{2\sqrt{a}}^{x}\left(\frac{x^2}{4}-a\right)^{1/2}dx. \tag{12}$$

a is found to be

$$a=\frac{2}{\pi}\left\{\frac{(\nu+\frac{1}{2})^2\kappa'^2-\mu}{\kappa'\sqrt{(\nu+\frac{1}{2})^2\kappa^2+\mu}}\times\left[\Pi\left(\frac{\pi}{2},\frac{\mu}{(\nu+\frac{1}{2})^2\kappa'^2},e\right)-K(e)\right]\right\}, \tag{13}$$

where

$$e=\frac{1}{\kappa'}\left(\frac{\mu}{(\nu+\frac{1}{2})^2\kappa^2+\mu}\right)^{1/2},\quad \mu>0,$$

and Π is the elliptic integral of the third kind. For $\mu<0$ a can be obtained from the above equation by replacing μ by $|\mu|$ and interchanging κ' and κ. Equation (13) has a power series expansion given by

$$a=\frac{1}{2\kappa\kappa'}\chi+\frac{\kappa^2-\kappa'^2}{16\kappa^3\kappa'^3(\nu+\frac{1}{2})}\chi^2+\frac{3-8\kappa^2\kappa'^2}{128\kappa^5\kappa'^5(\nu+\frac{1}{2})^2}\chi^3+O(\chi^4),$$

where

$$\chi=\frac{\mu}{\nu+\frac{1}{2}}.$$

If only the first term in the above expansion is retained,

$$a=\frac{\mu}{2\kappa\kappa'(\nu+\frac{1}{2})}. \tag{14}$$

If this value of a is used in Eqs. (11) and (12), near the turning points (small ϑ and φ) the above equations respectively yield

$$x=\sqrt{2\kappa/\kappa'}\sqrt{(\nu+\tfrac{1}{2})}\,\vartheta,\quad x=\sqrt{2\kappa'/\kappa}\sqrt{(\nu+\tfrac{1}{2})}\,\varphi. \tag{15}$$

Equation (9) is the desired differential equation. The asymptotic solution of this equation will be used to determine δ_ϑ and δ_φ.

A. The WKB phase constants δ_ϑ and δ_φ

The Weber equation

$$\frac{d^2y}{dx^2}+\left(\frac{x^2}{4}-a\right)y=0 \tag{16}$$

has solutions [5]

$$W(a,\pm x)=2^{-3/4}(\sqrt{G_1/G_3}\,y_e\mp\sqrt{2G_3/G_1}\,y_o), \tag{17}$$

where

$$y_e(x,a)=1+a\frac{x^2}{2!}+\left(a^2-\frac{1}{2}\right)\frac{x^4}{4!}+\left(a^3-\frac{7}{2}a\right)\frac{x^6}{6!}+\cdots, \tag{18}$$

$$y_o(x,a)=x+a\frac{x^3}{3!}+\left(a^2-\frac{3}{2}\right)\frac{x^5}{5!}+\left(a^3-\frac{13}{2}a\right)\frac{x^7}{7!}+\cdots, \tag{19}$$

and

$$G_1(a)=|\Gamma(\tfrac{1}{4}+\tfrac{1}{2}ia)|,$$

$$G_3(a)=|\Gamma(\tfrac{3}{4}+\tfrac{1}{2}ia)|.$$

For $x\gg|a|$ the Weber equation has asymptotic solutions [5]

$$W(a,x)=\left(\frac{2\eta(a)}{x}\right)^{1/2}\cos\left\{\frac{1}{4}x^2-a\,\ln x+\frac{\pi}{4}+\frac{1}{2}\phi_2(a)\right\}$$

and

$$W(a,-x)=\left(\frac{2}{\eta(a)x}\right)^{1/2}\sin\left\{\frac{1}{4}x^2-a\,\ln x+\frac{\pi}{4}+\frac{1}{2}\phi_2(a)\right\},$$

where

$$\eta(a)=\sqrt{1+e^{2\pi a}}-e^{\pi a},\quad \phi_2(a)=\arg\Gamma[\tfrac{1}{2}+ia].$$

From these two solutions we construct an even solution

$$\tilde{y}_e(x,a)=\sqrt[4]{2}\left(\frac{G_3(a)}{G_1(a)}\right)^{1/2}\left(\frac{1+\eta^2(a)}{\eta(a)}\right)^{1/2}\left(\frac{1}{x}\right)^{1/2}\times\cos\left\{\frac{x^2}{4}-a\,\ln x+\frac{\pi}{4}+\frac{1}{2}\phi_2(a)-\gamma(a)\right\},\tag{20}$$

and an odd solution

$$\tilde{y}_o(x,a)=-\sqrt[4]{2}\left(\frac{G_1(a)}{2G_3(a)}\right)^{1/2}\left(\frac{1+\eta^{2(a)}}{\eta(a)}\right)^{1/2}\left(\frac{1}{x}\right)^{1/2}\times\cos\left\{\frac{x^2}{4}-a\,\ln x+\frac{\pi}{4}+\frac{1}{2}\phi_2(a)+\gamma(a)\right\},\tag{21}$$

where

$$\tan\gamma(a)=\frac{1}{\eta(a)}=\sqrt{1+e^{2\pi a}}+e^{\pi a}.$$

In the region where φ is small so that the Lamé equation can be approximated by the Weber equation, yet ν is large so that the WKB solution for the Lamé equation is valid, and

$$x=\left(\frac{2\kappa'}{\kappa}\right)^{1/2}\left(\nu+\frac{1}{2}\right)^{1/2}\varphi\gg|a|,$$

both Eq. (5) and Eqs. (20) or (21) are solutions of the same differential equation. The phase constant δ_φ can therefore be determined by matching the phases of the two solutions. Depending on the prescribed boundary conditions, either $\tilde{y}_e$ or $\tilde{y}_o$ will be employed to obtain these phase constants. First consider the even solution for the Φ equation. From Eq. (12) we have

$$\begin{aligned}I_\varphi&=\int_{\varphi_t}^{\varphi}\left(\frac{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}{1-\kappa'^2\cos^2\varphi}\right)^{1/2}d\varphi\\&=\frac{1}{2}\int_{2\sqrt{a}}^{x}\sqrt{x^2-4a}\;dx\\&=\frac{x}{4}\sqrt{x^2-4a}-a\,\ln[x+\sqrt{x^2-4a}]+a\,\ln[2\sqrt{a}].\end{aligned}$$

Expanding the last expression in powers of a and keeping only terms of first order gives

$$I_\varphi=\frac{x^2}{4}-a\,\ln x+\frac{a}{2}\ln|a|-\frac{a}{2}.$$

The WKB solution, Eq. (5), thus becomes

$$\Phi(x)=\frac{A_\varphi\sqrt[4]{2}}{\sqrt[4]{(\nu+\frac{1}{2})\,\kappa\kappa'}\sqrt{x}}\times\cos\left\{\frac{x^2}{4}-a\,\ln x+\frac{a}{2}\ln|a|-\frac{a}{2}+\delta_\varphi\right\}.$$

Comparing the phase and amplitude of this equation with that of $\tilde{y}_e(x,a)$ we find

$$\delta_\varphi=\frac{\pi}{4}+\frac{1}{2}\phi_2(a)-\gamma(a)-\frac{a}{2}\ln|a|+\frac{a}{2}$$

and

$$A_\varphi=\sqrt[4]{\kappa\kappa'(\nu+\tfrac{1}{2})}\left(\frac{G_3(a)}{G_1(a)}\right)^{1/2}\left(\frac{1+\eta^2(a)}{\eta(a)}\right)^{1/2}.\tag{22}$$

It can be shown that

$$2\gamma(a)-\frac{3}{4}\pi=\arctan\left[\tanh\left(\frac{\pi a}{2}\right)\right]\equiv D(a)=-D(-a).$$

In terms of this new quantity we have

$$\delta_\varphi=-\frac{\pi}{8}+\frac{1}{2}\phi_2(a)-\frac{1}{2}D(a)-\frac{a}{2}\ln|a|+\frac{a}{2}.\tag{23}$$

The phase constant for the odd solution is obtained by comparing I_φ with the phase of Eq. (21). It is

$$\delta'_\varphi=\frac{5\pi}{8}+\frac{1}{2}\phi_2(a)+\frac{1}{2}D(a)-\frac{a}{2}\ln|a|+\frac{a}{2}.$$

A similar analysis gives

$$\delta_\vartheta=-\frac{\pi}{8}+\frac{1}{2}\phi_2(-a)+\frac{1}{2}D(a)+\frac{a}{2}\ln|a|-\frac{a}{2}\tag{24}$$

and

$$A_\vartheta=\sqrt[4]{(\nu+\tfrac{1}{2})\kappa\kappa'}\left(\frac{G_3(-a)}{G_1(-a)}\right)^{1/2}\left(\frac{1+\eta^2(-a)}{\eta(-a)}\right)^{1/2}.\tag{25}$$

And for the odd solution we find

$$\delta'_\vartheta=\frac{5\pi}{8}+\frac{1}{2}\phi_2(-a)-\frac{1}{2}D(a)+\frac{a}{2}\ln|a|-\frac{a}{2}.$$

For $\mu<0$ the role of the Φ and the Θ equations are interchanged. In other words, in this case the Θ equation is the one with the turning points. However, the expressions obtained for the phase constants still remain valid. In summary we have

$$\delta_\varphi=-\frac{\pi}{8}+\frac{1}{2}\phi_2(a)-\frac{1}{2}D(a)-\frac{a}{2}\ln|a|+\frac{a}{2},$$

$$\delta_\vartheta=-\frac{\pi}{8}-\frac{1}{2}\phi_2(a)+\frac{1}{2}D(a)+\frac{a}{2}\ln|a|-\frac{a}{2},$$

$$\delta'_\varphi=\frac{5\pi}{8}+\frac{1}{2}\phi_2(a)+\frac{1}{2}D(a)-\frac{a}{2}\ln|a|+\frac{a}{2},$$

$$\delta'_\vartheta=\frac{5\pi}{8}-\frac{1}{2}\phi_2(a)-\frac{1}{2}D(a)+\frac{a}{2}\ln|a|-\frac{a}{2}.$$

The above phase constants were derived for small values of a when the turning points lie close to 0 or π. For large positive values of a, $D(a)\to\pi/4$ and $\phi_2(a)\to a\ \ln|a|-a$. In this limit $\delta_\varphi=-\pi/4$, $\delta'_\varphi=3\pi/2$, $\delta_\vartheta=0$, and $\delta'_\vartheta=\pi/2$. For large negative values of a, $D(a)\to-\pi/4$ and $\phi_2(a)\to -|a|\ln|a|+|a|$. In this limit $\delta_\varphi=0$ $\delta'_\varphi=\pi/2$, $\delta_\vartheta=-\pi/4$, and $\delta'_\vartheta=3\pi/2$. These are the phase constants that would have been obtained had the problem been treated by the regular WKB method often employed in solving the Schrödinger equation in quantum mechanics. In solving the Schrödinger equation by the WKB method, the phase constants are obtained by matching the phase of the WKB solution with the phase of the asymptotic solution of the Airy's equation.

III. THE WKB SOLUTIONS

By determining the phase terms, δ_ϑ, δ_φ, δ'_ϑ, and δ'_φ we now have the complete WKB solutions of the Θ and the Φ equations. In this section we derive the WKB eigenvalue equations by applying the boundary conditions and imposing the requirement that the solutions for $(\vartheta;\varphi) > (\vartheta_t;\varphi_t)$ and those for $(\vartheta;\varphi) < (\pi-\vartheta_t;\pi-\varphi_t)$ join each other smoothly in their common region of validity. This results in relationships for the WKB eigenvalues. Both Dirichlet and Neumann boundary conditions will be considered.

A. Dirichlet boundary condition

For the Dirichlet boundary condition we have [1]

$$\Phi'(0)=0,\quad \Theta'(0)=0,$$
$$\Phi'(\pi)=0,\quad \Theta(\pi)=0.$$

That is, the Φ solution must be even at both $\varphi=0$ and $\varphi=\pi$, where the Θ solution must be even at $\vartheta=0$ and odd at $\vartheta=\pi$. The WKB solution of the Φ equation valid in the region $\varphi>\varphi_t$ is

$$\Phi^{<}(\varphi)=A_\varphi h(\varphi)\cos\left\{\int_{\varphi_t}^{\varphi}\left(\frac{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}{1-\kappa'^2\cos^2\varphi}\right)^{1/2}d\varphi+\delta_\varphi\right\},\quad \varphi>\varphi_t,$$

where

$$h(\varphi)\equiv\frac{1}{\sqrt[4]{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}}.$$

Since for the Dirichlet boundary condition $\Phi(\varphi)$ is even at both $\varphi=0$ and $\varphi=\pi$, the WKB solution for $\varphi<\pi-\varphi_t$ is given by

$$\Phi^{>}(\varphi)=A'_\varphi h(\varphi)\cos\left\{\int_{\varphi}^{\pi-\varphi_t}\left(\frac{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}{1-\kappa'^2\cos^2\varphi}\right)^{1/2}d\varphi+\delta_\varphi\right\},\quad \varphi<\pi-\varphi_t.$$

The two solutions, $\Phi^{<}(\varphi)$ and $\Phi^{>}(\varphi)$, must join smoothly in the region $\varphi_t<\varphi<\pi-\varphi_t$. Let us define

$$J_\varphi\equiv\int_{\varphi_t}^{\pi-\varphi_t}\left(\frac{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}{1-\kappa'^2\cos^2\varphi}\right)^{1/2}d\varphi$$
$$=2\int_{\varphi_t}^{\pi/2}\left(\frac{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}{1-\kappa'^2\cos^2\varphi}\right)^{1/2}d\varphi,\qquad(26)$$

and

$$b\equiv\int_{\varphi}^{\pi-\varphi_t}\left(\frac{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}{1-\kappa'^2\cos^2\varphi}\right)^{1/2}d\varphi+\delta_\varphi,$$

then

$$\int_{\varphi_t}^{\varphi}\left(\frac{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}{1-\kappa'^2\cos^2\varphi}\right)^{1/2}d\varphi$$
$$=\int_{\varphi_t}^{\pi-\varphi_t}\left(\frac{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}{1-\kappa'^2\cos^2\varphi}\right)^{1/2}d\varphi$$
$$-\int_{\varphi}^{\pi-\varphi_t}\left(\frac{(\nu+\frac{1}{2})^2\kappa'^2\sin^2\varphi-\mu}{1-\kappa'^2\cos^2\varphi}\right)^{1/2}d\varphi$$
$$=J_\varphi-b+\delta_\varphi.$$

In terms of these quantities we have

$$\Phi^{<}(\varphi)=A_\varphi h(\varphi)\cos\{J_\varphi-b+2\,\delta_\varphi\}$$

and

$$\Phi^{>}(\varphi)=A'_\varphi h(\varphi)\cos\ b.$$

Equating $\Phi^{<}$ and $\Phi^{>}$ gives

$$A_\varphi\{\cos\{J_\varphi+2\delta_\varphi\}\cos b+\sin\{J_\varphi+2\delta_\varphi\}\sin b\}=A'_\varphi\cos b.$$

The solution to this equation, which gives A'_φ, constant and independent of the parameter b, is obtained by setting

$$J_\varphi+2\delta_\varphi=m\pi,\quad m=0,1,\ldots.$$

Substituting for δ_φ, we find

$$J_\varphi+\phi_2(a)-D(a)-a\ln|a|+a=(m+\tfrac{1}{4})\pi,\quad m=0,1,\ldots.$$

For Dirichlet boundary condition $\Theta(\vartheta)$ is even around $\vartheta=0$ and odd around $\vartheta=\pi$. Then

$$\Theta^<(\vartheta)=A_\vartheta h(\vartheta)\cos\left\{\int_{\vartheta_t}^{\vartheta}\left(\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{1-\kappa^2\cos^2\vartheta}\right)^{1/2}d\vartheta+\delta_\vartheta\right\},\quad \vartheta>0,$$

where $h(\vartheta)$ is $h(\varphi)$ with the appropriate change of variables. The solution for $\vartheta>\pi+\vartheta_t$ is

$$\Theta^>(\vartheta)=A'_\vartheta h(\vartheta)\cos\left\{\int_{\pi+\vartheta_t}^{\vartheta}\left(\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{1-\kappa^2\cos^2\vartheta}\right)^{1/2}d\vartheta+\delta'_\vartheta\right\},\quad \vartheta>\pi+\vartheta_t.$$

Since this solution is odd with respect to $\vartheta=\pi$, the solution for $\vartheta<\pi-\vartheta_t$ is given by

$$\Theta^>(\vartheta)=-A'_\vartheta h(\vartheta)\cos\left\{\int_{\vartheta}^{\pi-\vartheta_t}\left(\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{1-\kappa^2\cos^2\vartheta}\right)^{1/2}d\vartheta+\delta'_\vartheta\right\},\quad \vartheta<\pi-\vartheta_t.$$

Requiring that $\Theta^<(\vartheta)$ and $\Theta^>(\vartheta)$ join each other smoothly results in

$$J_\vartheta+\delta_\vartheta+\delta'_\vartheta=n\pi,\quad n=1,2,\ldots.$$

Note that the reason n starts from 1 is to guarantee that

$$n\pi-\delta_\vartheta-\delta'_\vartheta>0,$$

since J_ϑ is positive. Substituting for δ_ϑ and δ'_ϑ we find

$$J_\vartheta+\phi_2(-a)+a\ln|a|-a=(n+\tfrac{1}{2})\pi,\quad n=0,1,\ldots.$$

For Dirichlet boundary condition we therefore have the set of eigenvalue equations

$$\begin{aligned}&J_\varphi+\phi_2(a)-D(a)-a\ln|a|+a=(m+\tfrac{1}{4})\pi,\quad m=0,1,\ldots,\\&J_\vartheta+\phi_2(-a)+a\ln|a|-a=(n+\tfrac{1}{2})\pi,\quad n=0,1,\ldots.\end{aligned}\tag{27}$$

The integrals defined by J_ϑ and J_φ can be expressed in terms of elliptic integrals [6]

$$J_\vartheta=\frac{2\mu}{\kappa'\sqrt{(\nu+\frac{1}{2})^2\kappa^2+\mu}}\left\{\Pi\left(\frac{\pi}{2},\frac{(\nu+\frac{1}{2})^2\kappa^2}{(\nu+\frac{1}{2})^2\kappa^2+\mu},r_+\right)\right\},$$

and

$$J_\varphi=\frac{2\mu}{\kappa'\sqrt{(\nu+\frac{1}{2})^2\kappa^2+\mu}}\times\left\{\Pi\left(\frac{\pi}{2},\frac{(\nu+\frac{1}{2})^2\kappa'^2-\mu}{(\nu+\frac{1}{2})^2\kappa'^2},r_+\right)-K(r_+)\right\}$$

for $\mu>0$ and

$$J_\vartheta=\frac{-2\mu}{\kappa\sqrt{(\nu+\frac{1}{2})^2\kappa'^2-\mu}}\times\left\{\Pi\left(\frac{\pi}{2},\frac{(\nu+\frac{1}{2})^2\kappa^2+\mu}{(\nu+\frac{1}{2})^2\kappa^2},r_-\right)-K(r_-)\right\},$$

and

$$J_\varphi=\frac{-2\mu}{\kappa\sqrt{(\nu+\frac{1}{2})^2\kappa'^2-\mu}}\left\{\Pi\left(\frac{\pi}{2},\frac{(\nu+\frac{1}{2})^2\kappa'^2}{(\nu+\frac{1}{2})^2\kappa'^2-\mu},r_-\right)\right\}$$

for $\mu<0$. In the above

$$r_+=\frac{\kappa}{\kappa'}\left(\frac{(\nu+\frac{1}{2})^2\kappa'^2-\mu}{(\nu+\frac{1}{2})^2\kappa^2+\mu}\right)^{1/2},$$

$$r_-=\frac{\kappa'}{\kappa}\left(\frac{(\nu+\frac{1}{2})^2\kappa^2-\mu}{(\nu+\frac{1}{2})^2\kappa'^2+\mu}\right)^{1/2},$$

Π is the elliptic integral of the third kind, and K is the elliptic integral of the first kind. We also have the following identity for J_ϑ and J_φ:

$$J_\vartheta+J_\varphi=(\nu+\tfrac{1}{2})\pi.\tag{28}$$

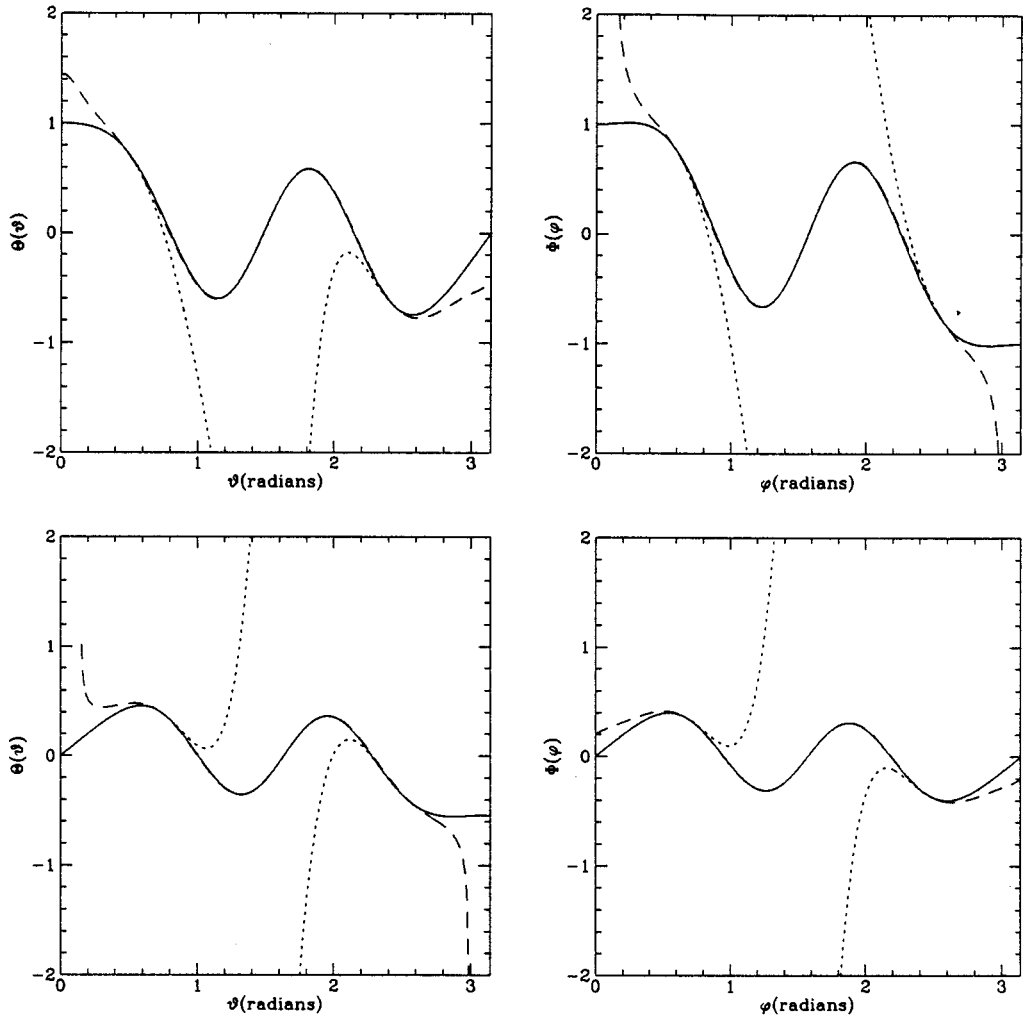

FIG. 2. The top two panels show the Θ and the Φ eigenfunctions for the exact (solid lines), the WKB (dashed lines) and the power series solution to the Weber equation (dotted lines) of the Lamé equations, subject to Dirichlet boundary condition. The corresponding exact eigenvalues are $(\nu,\mu)=(6.282\,285,0.442\,731)$, and the corresponding WKB eigenvalues are $(\nu,\mu)=(6.282\,324,0.441\,515)$. The bottom two panels show the same eigenfunctions for Neumann boundary condition with exact eigenvalues $(\nu,\mu)=(6.792\,665,-0.632\,047)$ and WKB eigenvalues $(\nu,\mu)=(6.792\,723,-0.630\,761)$. Note how the power series solutions overlay the exact solutions for ϑ and φ close to zero and π and connect smoothly with the WKB solutions.

B. Neumann boundary condition

For the Neumann boundary condition we have

$$\Phi(0)=0, \quad \Theta(0)=0,$$
$$\Phi(\pi)=0, \quad \Theta'(\pi)=0.$$

The Φ solution is thus odd with respect to $\varphi=0$ and $\varphi=\pi$ where the Θ solution is odd with respect to $\vartheta=0$ and even with respect to $\vartheta=\pi$. A similar approach yields the eigenvalue equations:

$$J_\varphi+\phi_2(a)+D(a)-a\,\ln|a|+a=(m+\tfrac{3}{4})\pi, \quad m=0,1,\ldots$$
$$J_\vartheta+\phi_2(-a)+a\,\ln|a|-a=(n+\tfrac{1}{2})\pi, \quad n=0,1,\ldots. \tag{29}$$

Note that Eqs. (27) and (29) are two parameter eigenvalue equations. This means that for a given value of n and m each set of equations must be solved simultaneously. We used the Newton-Raphson method in two dimensions to compute the eigenvalues, ν and μ for each boundary condition. The WKB eigenvalues agree remarkably well with the exact eigenvalues [1].

C. The WKB eigenfunctions

Once the eigenvalues are computed, the WKB eigenfunctions can be obtained from Eqs. (4) and (5). It is obvious that either of these solutions multiplied by a constant is still a solution. However, the amplitudes A_ϑ and A_φ were determined in such a way that the WKB solutions can be compared with the exact solution and with the solution of the Weber equation without the need to multiply them by a constant. This proves convenient when comparing individual eigenfunctions. The complete normal mode solution of the wave equation for this problem involves a sum over the product of all eigenfunctions divided by the normalization constant that removes any amplitude ambiguities [3]. The WKB normalization constant is derived later in this paper. For even solutions the amplitudes A_ϑ and A_φ given by Eqs.

(25) and (22) are determined such that $z(0)=1$ and $z'(0)=0$, where z can be either Θ or Φ. For odd solutions, on the other hand, the initial conditions are $z(0)=0$ and $z'(0)=1$. Because the derivative of the solution is nonzero at the initial point, A_ϑ and A_φ must be multiplied by the scale factors $d\vartheta/dx$ and $d\varphi/dx$, respectively. To see this, take, for example, $\Theta(\vartheta)$, which near $\vartheta=0$ reduces to the solution of the Weber equation, $y(x)$. Then

$$\left.\frac{d\Theta(\vartheta)}{d\vartheta}\right|_{\vartheta=0} \sim \left.\frac{dy(x)}{d\vartheta}\right|_{\vartheta=0} = \left.\frac{dy(x)}{dx}\right|_{x=0} \frac{dx}{d\vartheta} = 1.$$

Since $dy/dx|_{x=0}$ is chosen to be unity, $\Theta(\vartheta)$ multiplied by $d\vartheta/dx$ satisfies the above equation. The same is true for $\Phi(\varphi)$. $d\vartheta/dx$ and $d\varphi/dx$ may be obtained from Eq. (15).

As was pointed out before, we are particularly interested in those solutions for which the turning points lie close to 0 or π. In this case near 0 and π the solution can be obtained from the power series solution of the Weber equation given by Eqs. (18) and (19). For a given set of eigenvalues $\{\nu,\mu\}$, a is determined from Eq. (10). Then the power series solution are obtained by choosing ϑ or φ as independent variables and determining x from Eqs. (11) or (12), which is then used in Eqs. (18) and (19). The results are shown in Fig. 2. By only using a few terms, the power series solution overlays the exact solution for ϑ and φ close to zero and π and smoothly connects with the WKB solution in each case. In this way an excellent approximation to the exact solution can be obtained for the entire interval between 0 and π.

IV. SOLUTION FOR LARGE ν AND SMALL μ

It was pointed out earlier that the main contribution to the scattering amplitude comes from those eigenfunctions that correspond to small values of μ. It will be shown in this section that when $\mu/\nu^2 \ll 1$, the expressions for the eigenvalues and the integrals appearing in the phases of the WKB solutions can be approximated by simple algebraic functions. First, consider the integral

$$J_\vartheta = \int_0^\pi \left(\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{1-\kappa^2\cos\vartheta} \right)^{1/2} = \frac{2\kappa}{\kappa'}\left(\nu+\frac{1}{2}\right)\int_0^{\pi/2} \frac{\sqrt{\sin^2\vartheta+\alpha}}{\sqrt{1+\zeta\sin^2\vartheta}}\,d\vartheta,$$

where

$$\alpha = \frac{\mu}{\kappa^2(\nu+\frac{1}{2})^2} = \frac{2a\kappa'}{\kappa(\nu+\frac{1}{2})}, \quad \zeta = \left(\frac{\kappa}{\kappa'}\right)^2.$$

For small values of α this integral is approximated by [7]

$$J_\vartheta = 2(\nu+\tfrac{1}{2})\arctan\frac{\kappa}{\kappa'} + a\,\ln[8\kappa\kappa'(\nu+\tfrac{1}{2})] - a\,\ln|a| + a + O(\alpha^2). \tag{30}$$

Similarly,

$$J_\varphi = 2(\nu+\tfrac{1}{2})\arctan\frac{\kappa'}{\kappa} - a\,\ln[8\kappa\kappa'(\nu+\tfrac{1}{2})] + a\,\ln|a| - a + O(\alpha^2). \tag{31}$$

By adding the eigenvalue equations for the Dirichlet boundary condition, Eq. (27), and using Eq. (28) we find

$$\nu = m+n+\frac{1}{4}+\frac{D(a)}{\pi}. \tag{32}$$

For Neumann boundary condition we similarly find

$$\nu = m+n+\frac{3}{4}-\frac{D(a)}{\pi}. \tag{33}$$

By subtracting the eigenvalue equations for the Dirichlet boundary condition and using Eqs. (30), (31), and (32) we find

$$2\left(m+n+\frac{3}{4}+\frac{D(a)}{\pi}\right)\left(\frac{\pi}{2}-\beta\right)+2a\,\ln\left[8\kappa\kappa'\left(m+n+\frac{3}{4}+\frac{D(a)}{\pi}\right)\right]-2\phi_2(a)+D(a)=\left(n-m+\frac{1}{4}\right)\pi. \tag{34}$$

In the same way we find for the Neumann boundary condition

$$2\left(m+n+\frac{5}{4}-\frac{D(a)}{\pi}\right)\left(\frac{\pi}{2}-\beta\right)+2a\,\ln\left[8\kappa\kappa'\left(m+n+\frac{5}{4}-\frac{D(a)}{\pi}\right)\right]-2\phi_2(a)-D(a)=\left(n-m-\frac{1}{4}\right)\pi. \tag{35}$$

In the above equations use has been made of the relationships

$$\arctan\frac{\kappa'}{\kappa}=\frac{\beta}{2}, \quad \arctan\frac{\kappa}{\kappa'}=\left(\frac{\pi}{2}-\frac{\beta}{2}\right).$$

Equations (34) and (35) have the advantage that they only depend on the parameter a. This allows one to solve these equations for a by performing a search in one dimension (as opposed to two dimensions when the equations depend on ν as well) and then use Eqs. (32) and (33) to determine ν. The eigenvalues obtained in this way are still in excellent agreement with the exact eigenvalues. The phase of the WKB solution for the Θ equation, Eq. (4) is

$$f(\vartheta)=\int_{\vartheta_t}^{\vartheta}\left(\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{1-\kappa^2\cos^2\vartheta}\right)^{1/2}d\vartheta+\delta_\vartheta.$$

This can be written as

$$f(\vartheta)=f(\pi/2)-u(\vartheta), \tag{36}$$

where

$$f\left(\frac{\pi}{2}\right)=\int_{\vartheta_t}^{\pi/2}\left(\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{1-\kappa^2\cos^2\vartheta}\right)^{1/2}d\vartheta+\delta_\vartheta,$$

and

$$u(\vartheta)=\int_{\vartheta}^{\pi/2}\left(\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{1-\kappa^2\cos^2\vartheta}\right)d\vartheta$$
$$=\kappa\left(\nu+\frac{1}{2}\right)\int_{\vartheta}^{\pi/2}\frac{\sqrt{\sin^2\vartheta+\omega}}{\sqrt{1-\kappa^2\cos^2\vartheta}}\,d\vartheta,$$

where

$$\omega=\frac{\mu}{\kappa^2(\nu+\frac{1}{2})^2}.$$

Using the WKB eigenvalue equations and the expressions for δ_ϑ and δ_φ, we find for both Dirichlet and Neumann boundary conditions

$$f\left(\frac{\pi}{2}\right)=\frac{1}{2}\left(n+\frac{1}{4}\right)\pi+\frac{D(a)}{2}.$$

Similarly the phase of the Φ equation can be written as

$$g(\varphi)=g\left(\frac{\pi}{2}\right)-u(\varphi).$$

For the Dirichlet boundary condition we find

$$g\left(\frac{\pi}{2}\right)=\frac{m\pi}{2},$$

and for the Neumann boundary condition we find

$$g\left(\frac{\pi}{2}\right)=\frac{1}{2}\left(m+\frac{1}{2}\right)\pi-D(a).$$

In the above $g(\varphi)$ is $f(\vartheta)$ with κ replaced by κ', μ by $-\mu$, and δ_ϑ by δ_φ. For small ω it may be shown that [7]

$$u(\vartheta)=\left(\nu+\frac{1}{2}\right)\arcsin[\kappa\,\cos\vartheta]$$
$$+a\,\ln\left|\frac{\sqrt{1-\kappa^2\cos^2\vartheta}+\kappa'\cos\vartheta}{\sin\vartheta}\right|, \tag{37}$$

where $a=\mu/2\kappa\kappa'(\nu+\frac{1}{2})$. A similar solution can be obtained for $u(\varphi)$ by replacing κ with κ' and a with $-a$.

V. THE WKB NORMALIZATION

In this section we derive an approximate formula for the WKB normalization coefficient that is valid for small values of μ and large values of ν. The normalization coefficent for the solution of the wave equation in the sphero-conal coordinate system is given by [2]

$$N=\int_0^\pi\int_0^\pi|\Theta_{\text{Lamé}}(\vartheta)\Phi_{\text{Lamé}}(\varphi)|^2\sigma d\vartheta\,d\varphi, \tag{38}$$

where

$$\sigma=\frac{\kappa^2\sin^2\vartheta+\kappa'^2\sin^2\varphi}{\sqrt{(1-\kappa^2\cos^2\vartheta)(1-\kappa'^2\cos^2\varphi)}}.$$

Let us define

$$\chi_{\text{Lamé}}(\vartheta)\equiv\Theta^2_{\text{Lamé}}(\vartheta)\quad\text{and}\quad\chi_{\text{Lamé}}(\varphi)\equiv\Phi^2_{\text{Lamé}}(\varphi),$$

then by submitting these quantities in Eq. (38), we find

$$N=\int_0^\pi\chi_{\text{Lamé}}(\vartheta)\frac{\kappa^2\sin^2\vartheta}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta$$
$$\times\int_0^\pi\chi_{\text{Lamé}}(\varphi)\frac{1}{\sqrt{1-\kappa'^2\cos^2\varphi}}d\varphi$$
$$+\int_0^\pi\chi_{\text{Lamé}}(\varphi)\frac{\kappa'^2\sin^2\varphi}{\sqrt{1-\kappa'^2\cos^2\varphi}}d\varphi$$
$$\times\int_0^\pi\chi_{\text{Lamé}}(\vartheta)\frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta. \tag{39}$$

The above integrals are of the general form

$$\int\chi_{\text{Lamé}}(\phi)v(\phi)d\phi,$$

where ϕ can be either ϑ or φ and $v(\phi)$ represents the other functions appearing in Eq. (39). Now let us write $\int_0^\pi\chi_{\text{Lamé}}(\vartheta)v(\vartheta)d\vartheta$ as

$$\int_0^\pi\chi_{\text{Lamé}}(\vartheta)v(\vartheta)d\vartheta=\int_0^\pi[\chi_{\text{Lamé}}(\vartheta)-\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)]v(\vartheta)d\vartheta$$
$$+\int_0^\pi\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)v(\vartheta)d\vartheta, \tag{40}$$

where $\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)$ is the WKB solution of the Lamé equation. The first integral can be written as

$$\int_0^{\pi}[\chi_{\text{Lamé}}(\vartheta)-\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)]\upsilon(\vartheta)d\vartheta$$
$$=\int_0^{\Delta}[\chi_{\text{Lamé}}(\vartheta)-\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)]\upsilon(\vartheta)d\vartheta$$
$$+\int_{\Delta}^{\pi-\Delta}[\chi_{\text{Lamé}}(\vartheta)-\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)]\upsilon(\vartheta)d\vartheta$$
$$+\int_{\pi-\Delta}^{\pi}[\chi_{\text{Lamé}}(\vartheta)-\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)]\upsilon(\vartheta)d\vartheta.$$

Since for ϑ close to 0 or π the Lamé equation reduces to the Weber equation, Δ is chosen to be small enough such that for $0<\vartheta<\Delta$ and $\pi-\Delta<\vartheta<\pi$, the solution of the Lamé equation can be represented by the solution of the Weber equation, denoted by χ_{Weber}, and the WKB solution of the Lamé equation can be represented by the WKB solution of the Weber equation, denoted by $\chi^{\text{WKB}}_{\text{Weber}}$, and given by Eqs. (20) or (21). In the region $\Delta<\vartheta<\pi-\Delta$, both the solution of the Lamé equation and the solution of the Weber equation can be represented by their WKB solutions:

$$\chi_{\text{Lamé}}(\vartheta)=\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta),\quad \chi_{\text{Weber}(\vartheta)}=\chi^{\text{WKB}}_{\text{Weber}}(\vartheta).$$

Based on this argument the middle integral in the above equation is zero and in the other integrals $\chi_{\text{Lamé}}$ can be replaced by χ_{Weber}, resulting in

$$\int_0^{\pi}[\chi_{\text{Lamé}}(\vartheta)-\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)]\upsilon(\vartheta)d\vartheta$$
$$=\int_0^{\Delta}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\upsilon(\vartheta)d\vartheta$$
$$+\int_{\pi-\Delta}^{\pi}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\upsilon(\vartheta)d\vartheta.$$

Substituting these in Eq. (40) we find

$$\int_0^{\pi}\chi_{\text{Lamé}}(\vartheta)\upsilon(\vartheta)d\vartheta$$
$$=\int_0^{\Delta}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\upsilon(\vartheta)d\vartheta$$
$$+\int_{\pi-\Delta}^{\pi}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\upsilon(\vartheta)d\vartheta$$
$$+\int_0^{\pi}\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)\upsilon(\vartheta)d\vartheta.$$

In the second integral in the above equation let $\vartheta'=\pi-\vartheta$, so

$$\int_{\pi-\Delta}^{\pi}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\upsilon(\vartheta)d\vartheta$$
$$=\int_0^{\Delta}[\chi_{\text{Weber}}(\vartheta')-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta')]\upsilon(\vartheta')d\vartheta'.$$

Next, let us write

$$\int_0^{\Delta}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\upsilon(\vartheta)d\vartheta$$
$$=\int_0^{\infty}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\upsilon(\vartheta)d\vartheta$$
$$-\int_{\Delta}^{\infty}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\upsilon(\vartheta)d\vartheta.$$

For $\vartheta>\Delta$, $\chi_{\text{Weber}}(\vartheta)=\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)$ and thus the second integral in the above equation is zero and

$$\int_0^{\Delta}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\upsilon(\vartheta)d\vartheta$$
$$=\int_0^{\infty}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\upsilon(\vartheta)d\vartheta.$$

Thus

$$\int_0^{\pi}\chi_{\text{Lamé}}(\vartheta)\upsilon(\vartheta)d\vartheta$$
$$=\int_0^{\infty}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\upsilon(\vartheta)d\vartheta$$
$$+\int_0^{\infty}[\chi_{\text{Weber}}(\vartheta')-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta')]\upsilon(\vartheta)'d\vartheta'$$
$$+\int_0^{\pi}\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)\upsilon(\vartheta)d\vartheta. \tag{41}$$

Note that in the first integral the functions in the square brackets are valid near $\vartheta=0$ and in the second integral they are valid near $\vartheta=\pi$. In the equation for the normalization we either have

$$\upsilon(\vartheta)=\frac{\kappa^2\sin^2\vartheta}{\sqrt{1-\kappa^2\cos^2\vartheta}}$$

or

$$\upsilon(\vartheta)=\frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta}}.$$

In the first case we have

$$\int_0^{\pi}\chi_{\text{Lamé}}(\vartheta)\frac{\kappa^2\sin^2\vartheta}{\sqrt{1-\kappa^2\cos^2}}d\vartheta$$
$$=\int_0^{\infty}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\frac{\kappa^2\sin^2\vartheta}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta$$
$$+\int_0^{\infty}[\chi_{\text{Weber}}(\vartheta')-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta')]\frac{\kappa^2\sin^2(\vartheta')}{\sqrt{1-\kappa^2\cos^2\vartheta'}}d\vartheta'$$
$$+\int_0^{\pi}\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)\frac{\kappa^2\sin^2\vartheta}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta.$$

There is negligible contribution from the first two integrals on the right hand side because for small values of ϑ, $\sin\vartheta \approx 0$ and for large values of ϑ, $\chi_{\text{Weber}}(\vartheta) = \chi^{\text{WKB}}_{\text{Weber}}(\vartheta)$. Therefore, we find

$$\int_0^{\pi} \chi_{\text{Lamé}}(\vartheta)\frac{\kappa^2\sin^2\vartheta}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta = \int_0^{\pi} \chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)\frac{\kappa^2\sin^2\vartheta}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta.$$

In the second case

$$\begin{aligned}\int_0^{\pi} \chi_{\text{Lamé}}(\vartheta)\frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta &= \int_0^{\infty}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]\frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta \\ &+ \int_0^{\infty}[\chi_{\text{Weber}}(\vartheta')-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta')]\frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta'}}d\vartheta' \\ &+ \int_0^{\pi} \chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)\frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta.\end{aligned}$$

Here again there is negligible contribution from the first two integrals for large values of ϑ since $\chi_{\text{Weber}}(\vartheta)=\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)$. For small valuers of ϑ

$$\frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta}} \approx \frac{1}{\kappa'},$$

and thus we have

$$\begin{aligned}\int_0^{\pi} \chi_{\text{Lamé}}(\vartheta)\frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta &= \frac{1}{\kappa'}\int_0^{\infty}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]d\vartheta \\ &+ \frac{1}{\kappa'}\int_0^{\infty}[\chi_{\text{Weber}}(\vartheta')-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta')]d\vartheta' \\ &+ \int_0^{\pi} \chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)\frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta.\end{aligned}$$

Let us define

$$\epsilon_{\vartheta} = \int_0^{\infty}[\chi_{\text{Weber}}(\vartheta)-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta)]d\vartheta,$$

and

$$\epsilon'_{\vartheta} = \int_0^{\infty}[\chi_{\text{Weber}}(\vartheta')-\chi^{\text{WKB}}_{\text{Weber}}(\vartheta')]d\vartheta',$$

then

TABLE I. Comparison between the exact and WKB normalization coefficients.

(n,m)	α_{exact}	α_{WKB}	N_{exact}	N_{WKB}
(1,1)	0.465 255	0.324 84	3.8244	3.3266
(2,1)	−0.452 788	−0.348 023	2.9522	3.0724
(2,2)	0.530 210	0.504 450	5.0025	4.8042
(3,1)	−0.549 215	−0.537 476	8.4871	8.7563
(3,2)	0.057 476	0.055 747	1.3784	1.3586
(3,3)	0.617 823	0.611 585	19.0350	18.5510
(4,1)	−0.652 852	−0.651 458	41.9459	41.1747
(4,2)	−0.126 564	−0.124 423	1.4276	1.4592
(4,3)	0.204 048	0.201 308	2.2521	2.2066
(4,4)	0.691 944	0.694 140	98.7856	93.2777
(5,1)	−0.720 185	−0.725 213	229.3953	211.8818
(5,2)	−0.251 494	−0.244 911	3.7883	3.8674
(5,3)	0.029 370	0.029 055	0.8805	0.8758
(5.4)	0.309 851	0.308 281	7.8471	7.7465
(5,5)	0.745 515	0.752 932	539.3419	480.6180

$$\int_0^{\pi} \chi_{\text{Lamé}}(\vartheta)\frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta = \frac{\epsilon_{\vartheta}+\epsilon'_{\vartheta}}{\kappa'} + \int_0^{\pi} \chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)\frac{1}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta.$$

Following a similar procedure using integrals involving the variable φ and substituting in Eq. (38) yields

$$\begin{aligned}N = &\int_0^{\pi}\frac{\kappa^2\sin^2(\vartheta)\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta \\ &\times\left[C_{\varphi}\frac{\epsilon_{\varphi}+\epsilon'_{\varphi}}{\kappa} + \int_0^{\pi}\frac{\chi^{\text{WKB}}_{\text{Lamé}}(\varphi)}{\sqrt{1-\kappa'^2\cos^2\varphi}}d\varphi\right] \\ &+ \int_0^{\pi}\frac{\kappa'^2\sin^2(\varphi)\chi^{\text{WKB}}_{\text{Lamé}}(\varphi)}{\sqrt{1-\kappa'^2\cos^2\varphi}}d\varphi \\ &\times\left[C_{\vartheta}\frac{\epsilon_{\vartheta}+\epsilon'_{\vartheta}}{\kappa'} + \int_0^{\pi}\frac{\chi^{\text{WKB}}_{\text{Lamé}}(\vartheta)}{\sqrt{1-\kappa^2\cos^2\vartheta}}d\vartheta\right], \qquad (42)\end{aligned}$$

where

$$\epsilon_{\varphi} = \int_0^{\infty}[\chi_{\text{Weber}}(\varphi)-\chi^{\text{WKB}}_{\text{Weber}}(\varphi)]d\varphi,$$

and

$$\epsilon'_{\varphi} = \int_0^{\infty}[\chi_{\text{Weber}}(\varphi')-\chi^{\text{WKB}}_{\text{Weber}}(\varphi')]d\varphi',$$

and C_{ϑ} and C_{φ} are proportionality constants given by

$$\int \psi_{\text{Lamé}}(\vartheta)^2 d\vartheta = C_{\vartheta}\int \psi^2_{\text{Weber}}(x)dx, \qquad (43)$$

and

$$\int \psi_{\text{Lamé}}(\varphi)^2 d\varphi = C_\varphi \int \psi^2_{\text{Weber}}(x)dx. \tag{44}$$

Expressions for C_ϑ and C_φ are derived in the Appendix. The evaluation of the integrals appearing in the expressions for ϵ's is quite lengthy. In this paper we only report the results. The interested reader may refer to Ref. [7] for a detailed derivation. We find the WKB normalization coefficient

$$N = \frac{A_\vartheta^2 A_\varphi^2 \pi}{2(\nu+\frac{1}{2})^2 \kappa\kappa'}\left[\frac{1}{2}\ln\left\{8\kappa\kappa'\left(\nu+\frac{1}{2}\right)\right\} - \frac{1}{2}\mathrm{Re}\left\{\psi\left(\frac{1}{2}+ia\right)\right\} \pm \frac{\arcsin\kappa}{e^{\pi a}+e^{-\pi a}}\right], \tag{45}$$

where ψ is the digamma function defined by $\psi(z) = d\Gamma(z)/dz$ In the above equation the plus sign is used for the Dirichlet boundary condition and the minus sign is used for the Neumann boundary condition. Table I shows a comparison between the exact and the WKB normalization coefficients for a 90° PAS subject to Dirichlet boundary condition. In Table I

$$\alpha_{\text{exact}} = \frac{\mu_{\text{exact}}}{(\nu_{\text{exact}}+\frac{1}{2})^2\kappa^2}, \quad \alpha_{\text{WKB}} = \frac{\mu_{\text{WKB}}}{(\nu_{\text{WKB}}+\frac{1}{2})^2\kappa^2},$$

and n and m are defined in Eq. (27). As can be seen from Table I, for large values of n, where the WKB solution is accurate, and small α, where the approximation leading to Eq. (45) is valid, N_{WKB} matches N_{exact} very closely.

APPENDIX: THE DERIVATION OF C_ϑ and C_φ

We use the fact that for small angles the Lamé and the Weber equations have the same solution up to a multiplicative constant. First, note that the WKB solution of the Weber equation using the amplitude factors in Eqs. (20) and (21) in Eq. (17) is given by

$$\psi_{Weber}(x) = \sqrt{2}\left(\frac{1+\eta^2(-a)}{\eta(-a)}\right)^{1/2} \frac{1}{\sqrt[4]{x^2+4a}} \times \cos\left(\frac{\int_0^x \sqrt{x^2+4a}}{2}dx\right).$$

Then

$$\int \psi_{\text{Weber}}(x)^2 dx = 2\frac{1+\eta^2(-a)}{\eta(-a)}\int \frac{1}{\sqrt{x^2+4a}} \times \cos^2\left(\frac{1}{2}\int_0^x \sqrt{x^2+4a}dx\right)dx. \tag{A1}$$

On the other hand we have

$$\int \psi^2_{\text{Lamé}}(\vartheta)d\vartheta = A_\vartheta^2 \int \frac{\cos^2\left(\int_0^\vartheta \left[\frac{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}{1-\kappa^2\cos^2\vartheta}\right]^{1/2} d\vartheta\right)}{\sqrt{(\nu+\frac{1}{2})^2\kappa^2\sin^2\vartheta+\mu}}d\vartheta.$$

The arguments of the cosines in the above two equations are the same according to Eq. (11). For small ϑ, we replace $\sin\vartheta$ by ϑ and use

$$\vartheta = \left(\frac{\kappa'}{2\kappa}\right)^{1/2}\frac{x}{\sqrt{(\nu+\frac{1}{2})}},$$

to get

$$\int \psi^2_{\text{Lamé}}(\vartheta)d\vartheta = \frac{A_\vartheta^2}{(\nu+\frac{1}{2})\kappa}\int \frac{1}{\sqrt{x^2+4a}}\cos^2\left(\frac{1}{2}\int_0^x \sqrt{x^2+4a\,dx}\right)dx. \tag{A2}$$

Comparing Eqs. (A1) and (A2) with Eq. (43) yields

$$C_\vartheta = \frac{A_\vartheta^2}{2\kappa(\nu+\frac{1}{2})}\frac{\eta(-a)}{1+\eta^2(-a)},$$

where $a = \mu/2\kappa\kappa'(\nu+\frac{1}{2})$. In a similar manner we find

$$C_\varphi = \frac{A_\varphi^2}{2\kappa'(\nu+\frac{1}{2})}\frac{\eta(a)}{1+\eta^2(a)}.$$

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